Comparative Analysis of the Generalized Finite Element Method and the Ultra Weak Variational Formulation Approaches to Solve Electromagnetic Scattering Problems

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This paper presents a comparative analysis of the Ultra Weak Variational Formulation (UWVF) and the Generalized Finite Element Method (GFEM) approaches to compute the solution field generated by the scattering of an incident plane wave by a PEC obstacle. The aim is to show how the specific features of each approach can be used to improve the approximated solution. The results are supported by numerical experiment and a case study for a single simple geometry.

***Index Terms* — Ultra Weak Variational Formulation, Generalized Finite Element Method, Plane Wave Basis, Condition Number.**

I. Introduction

For Helmholtz equation the classical Finite Element Method (FEM) requires the mesh size parameter h smaller than 1/10th of the wavelength. This requirement may be an issue when the physical system is defined in an electrically large structure [1]. In this case, not only the computational cost will be high but the accuracy can also degrade. To overcome these drawbacks new versions of the FEM have been proposed in the literature. The main feature of these methods is that unlike the standard FEM the trial space might not be polynomials [2].

Here we consider two alternate approaches of the FEM, the Generalized Finite Element Method (GFEM) [3] and the Ultra Weak Variational Formulation (UWVF) [4] to solve the Helmholtz equation. The common feature is that they include plane wave basis functions to approximate the solution. It leads to accurate solution with mesh size greater than one wavelength [5]. On the other hand, while the GFEM is a continuous Galerkin method the UWVF is a discontinuous one. This distinction cause them to perform numerically different.

In this paper we compare the GFEM and the UWVF approaches for solving electromagnetic scattering problems. The aim is to show how they can be used to improve the approximated solution. The results are supported by numerical experiments and a case study.

II. Description of the approaches

The model problem solves the electromagnetic field generated by the scattering of an incident plane wave by a PEC obstacle. The governing equation is the Helmholtz equation $\nabla^2 u + k^2 u = 0$ where k is the wave number and u is the z component of the electric field. On the boundary of the obstacle we impose homogeneous Dirichlet. The exterior domain is truncated with a zero order absorbing boundary condition.

A. GFEM Approach

The key feature of the GFEM is the enrichment of the standard FEM by including solutions of the homogeneous governing equation. In this work plane waves are used for the Helmholtz equation. On each triangular element, the scalar field is expanded as in [6].

$$u_h(\boldsymbol{x}) = \sum_{i=1}^{3} N_i(\boldsymbol{x}) \left(u_i + \sum_{q=1}^{Q} a_{iq} \left(\psi_{iq}(\boldsymbol{x}) - \psi_q(\boldsymbol{x}_i) \right) \right) \quad (1)$$

where $N_i(\boldsymbol{x})$ are the hat FEM function and a_{iq} are the plane wave amplitudes at the q direction. The functions $\psi_{iq}(\boldsymbol{xi})$ and the coefficients u_i are associated with each node. Their presence makes it easy to impose essential boundary conditions at the expense of adding one unknown per node [6]. The main role of the partition of the unity is to glue-together the element-wise approximation. The number of plane waves attached to each node will depend on the wave number and element size.

The stiffness matrix of GFEM is built using the Galerkin scheme to discretize the standard weak form of the problem. The weight functions are chosen from plane wave basis. The algebraic system is symmetric and sparse.

B. UWVF Approach

Unlike GFEM, the UWVF is based on a discontinuous Galerkin FEM. For a given partition of the domain we restrict the strong form to each element with homogeneous physical parameters. On the interfaces between elements the continuity of the solution and its normal derivative are enforced via Robin type condition. Using test functions that are solutions of the adjoint Helmholtz equation, the variational equations for UWVF are obtained from integration by parts of the element strong form [4]. Two consequences of this formulation are that each element has its own Degrees of Freedom (DoF) and the continuity of the approximate solution is weakly enforced. To discretize the variational form the approximate solution and the weighting functions are represented by linear combination of the plane waves defined in each element. The resulting matrix equation can be written as $(D - C)X = b$, where D is Hermitian block diagonal matrix. The matrix C has off-diagonal block structure and represents the interface constraints. Since it is cheap to obtain the inverse of D, we solve the preconditioned form $(I - D^{-1}C)X = D^{-1}b$ as suggested in [5]. Once the solution vector X is computed, the field inside each element is approximated as

$$u_h(\boldsymbol{x}) = \sum_{q=1}^{Q} x_{iq}\psi_{iq}(\boldsymbol{x}). \tag{2}$$

III. Numerical Experiments

In the simulations the boundary of the obstacle is a circle of radius 0.3λ. The truncating boundary defines a circle of radius 3λ. The wave number is $k = 20\pi$ and the direction of the incident wave is parallel to x axis. The partition of the computational domain is done by triangular elements with $h_{max} = 1\lambda$, where h_{max} is the longest edge. Close to the scatterer, there is a concentration of small elements in order to represent the circular shape of the object. The mesh has 184 nodes and 332 elements. The accuracy of the two methods is compared in Fig. 1a. The relative error in the solution is plotted against the number of DoF. For this case, it is clear that the GFEM performs better than UWVF when we increase the number of DoF.

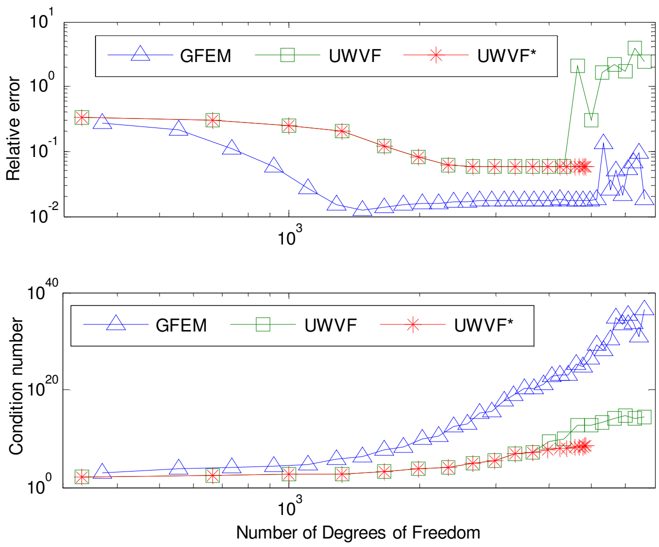

Fig. 1. Numerical performance of the GFEM, UWVF and UWVF*, a) relative error in the solution and b) condition number of the linear system.

As can be seen in Fig. 1b both approaches lead to ill-conditioning of the linear system. This potentially results in degradation of the solution when we increase the number of wave directions, Fig. 1a. Next we try a non-uniform distribution of wave directions for each element in the UWVF. In this case, the condition number of the element D matrix is verified when new directions are attached [7]. In this study the number of basis functions for each element is chosen such that the condition number stay below the limit 10^{10}. This approach is called UWVF*. Note that the unstable behavior for high condition number is mitigated, however, this technique does not guarantee a lower bound on the accuracy.

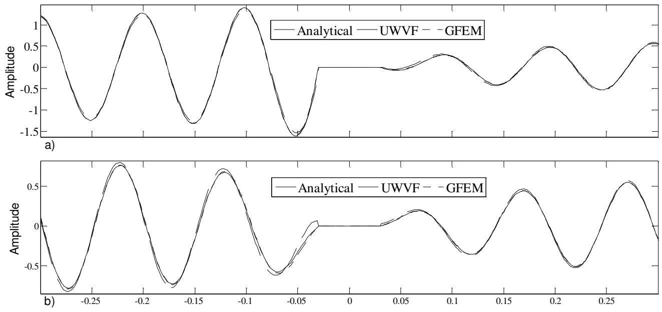

Fig. 2. The total field obtained from GFEM, UWVF and Analytical solutions sampled along the *x* axis, a) real part and b) imaginary part.

The total field solution along the x axis of the computational domain is shown in Fig. 2. For the GFEM the relative error is 1.25% and for the UWVF 5.81%. They correspond to the smaller error shown in Fig. 1a. No dispersion error is noticeable, however, the UWVF solution presents a significant amplitude error close to the scatterer.

IV. Conclusion

A case study was used to access the numerical performance of the GFEM and UWVF in solving the electromagnetic scattering with large wave number. Both approaches lead to ill-conditioning of the linear system, however, unlike the GFEM, the UWVF allows to control the condition number through the element D matrix. In terms of accuracy the GFEM performed better than UWVF but unstable behavior can be observed. In the full paper, more realistic case studies and new experiments will be investigated.

V. Acknowledgement

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VI. References

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